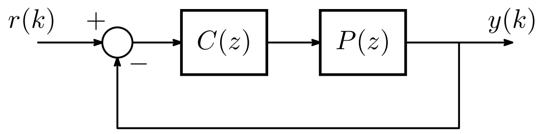
Exercise 2. Consider the feedback interconnection



where

$$P(z) = \frac{1}{z^2+1}.$$

By resorting to the direct synthesis method based on Diophantine equations, find a digital controller $C(z)$ and the value p such that

- the closed loop transfer function $W(z)$ has all poles in p, where $-1 < p < 1$;
- the obtained controller $C(z)$ is able to track *asymptotically* the step reference signal.

Solution. The denominator of $W(z)$ is $D_*(z) = (z-p)^\mu$ to satisfy the first requirement. The controller $C(z)$, instead, must have the form

$$C(z) \triangleq \frac{Y(z)}{X(z)} = \frac{Y(z)}{(z-1)X_1(z)},$$

in order to be able to track the step reference signal. Now, we can write the Diophantine equation of $W(z)$'s denominator:

$$D_*(z) = (z-p)^\mu = (z^2+1)(z-1)X_1(z) + Y(z). \qquad (*)$$

The simplest choice is to pick solutions with degree zero, $X_1(z) = x_0$ and $Y(z) = y_0$, so that $\mu = 3$ and the Diophantine equation $(*)$ becomes

$$\begin{aligned} z^3 - 3pz^2 + 3p^2z - p^3 &= x_0(z^2+1)(z-1) + y_0 = \\ &= x_0z^3 - x_0z^2 + x_0z + (y_0 - x_0). \end{aligned}$$

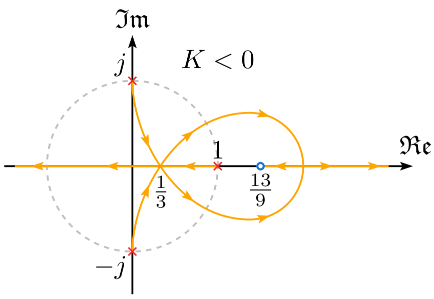
However, this choice does not allow to satisfy the condition on $D_*(z)$, in fact the four-equation system for coefficients has no solution for x_0, y_0, p. Increasing the degree of $Y(z) = y_0 + y_1z$ gives an additional degree of freedom while the system has four equations as well:

$$\begin{aligned} &(z-p)^3 = x_0(z^2+1)(z-1) + y_0 + y_1z \iff \\ &\iff z^3 - 3pz^2 + 3p^2z - p^3 = x_0z^3 - x_0z^2 + (x_0+y_1)z + (y_0 - x_0) \iff \\ &\iff \begin{cases} x_0 = 1 \\ x_0 = 3p \\ x_0 + y_1 = 3p^2 \\ x_0 - y_0 = p^3 \end{cases} \implies \begin{cases} p = \frac{1}{3} \\ x_0 = 1 \\ y_0 = \frac{26}{27} \\ y_1 = -\frac{2}{3}. \end{cases} \end{aligned}$$

Therefore, the poles of $W(z)$ are placed in $p = \frac{1}{3}$ and the controller is

$$C(z) = \frac{\frac{26}{27} - \frac{2}{3}z}{z-1} = -\frac{2}{3} \cdot \frac{z - \frac{13}{9}}{z-1}. \qquad \blacksquare$$

Observation. Letting $K = \mathbb{K}_{\mathrm{B}}[C(z)]$, i. e. $C(z) = K \cdot \frac{z-13/9}{z-1}$, the negative root locus of $W(z)$'s denominator exhibits a triple point in $p = \frac{1}{3}$ for $K = -\frac{2}{3}$ as expected, because all the three poles of $W(z)$ are placed in p:



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Question on theory. Recall the conditions ensuring that in a feedback control scheme a reference signal can be tracked with zero asymptotic error and show how these conditions are obtained.

Internal model principle. A stable feedback interconnection is capable of *asymptotic* tracking a reference signal $r(k)$ if the unstable poles of $R(z) \triangleq \mathscr{Z}[r(k)]$ are also poles of the open-loop transfer function $T(z) \triangleq C(z)P(z)$ with at least the same multiplicity.

WARNING: this proof is different from the one proposed by the adopted book (*Lecture Notes on Digital Control* by G. Baggio, M. Bisiacco, A. Ferrante, F. Ticozzi, S. Zampieri). I do not guarantee it is fully equivalent to it, or even correct. Use it to design your multi-loop curvature motor digital controller at your own risk. Contact me if you can improve it.

Proof. In a feedback control scheme, we have asymptotic tracking when the error $e(k)$ approaches 0 for $k \to \infty$; therefore, for the final value theorem,

$$0 = \lim_{k\to\infty} e(k) = \lim_{z\to 1} (1 - z^{-1})E(z)$$

where $E(z) \triangleq \mathscr{Z}[e(k)]$. Since $E = R - TE$,

$$E = \frac{R}{1+T}.$$

Now, let $R(z) \triangleq \frac{N_R(z)}{D_R(z)}$ and $T(z) \triangleq \frac{N_T(z)}{D_T(z)}$; so, we require that

$$\lim_{z\to 1} \frac{z-1}{z} \cdot \frac{N_R}{D_R} \cdot \frac{D_T}{D_T + N_T} = 0.$$

The only way to make the limit go to zero is that the $z - 1$ term is not simplified by D_R and $N_T + D_T$; however, since the feedback interconnection is stable, also $\frac{T}{1+T}$ must be stable and so $N_T + D_T$ cannot have a pole in $z = 1$.

In the case $R(z)$ has one or more poles in $z = 1$, the open-loop transfer function $T(z)$ must include them (with at least the same multiplicity) in its denominator D_T in order to simplify D_R and guarantee the limit is zero.

If all unstable poles of $R(z)$ are included in $T(z)$ with at least the same multiplicity, then for sure all poles in $z = 1$ of $R(z)$ are included in $T(z)$ with at least the same multiplicity and the asymptotic error when tracking $r(k)$ is zero. ■

Updated 2022-08-02