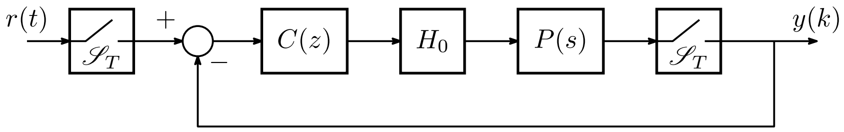
Exercise 1. Consider the scheme represented in the figure, where

$$P(s) = \frac{1}{s-1},$$

$\mathscr{S}_T$ is the sampler with sampling period T and H_0 is the zero-order holder.



1. Find, if possible, a digital controller $C(z) = \frac{K}{z-p}$, where $K, p \in \mathbb{R}$, that internally stabilizes the system and such that the output is able to track *asymptotically* the step reference signal.
2. Find, if possible, a digital controller with relative degree zero that internally stabilizes the system and such that the output is able to *deadbeat* track the step reference signal.
3. Assume now that $T = \frac{\pi}{3}$. Find, if possible, a digital controller with relative degree zero that internally stabilizes the system and such that the output is able to track *asymptotically* the reference signal $r(t) = \cos t$.

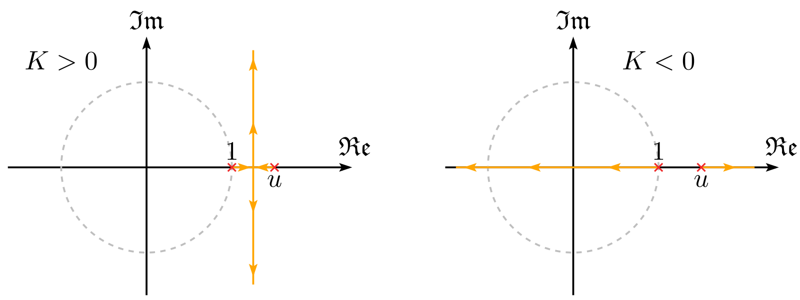
Solution. First of all, we calculate the discretized $\tilde{P}(z)$ by sample and hold:

$$\begin{aligned}
\tilde{P}(z) &= \frac{z-1}{z}\mathscr{Z}\left[\mathscr{S}_T\left[\mathscr{L}^{-1}\left[\frac{P(s)}{s}\right]\right]\right] = \\
&= \frac{z-1}{z}\mathscr{Z}\left[\mathscr{S}_T\left[\mathscr{L}^{-1}\left[-\frac{1}{s}+\frac{1}{s-1}\right]\right]\right] = \\
&= \frac{z-1}{z}\mathscr{Z}\left[\mathscr{S}_T\left[-1+e^t\right]\right] = \\
&= \frac{z-1}{z}\mathscr{Z}\left[-1+(e^T)^k\right] = \\
&= \frac{z-1}{z}\left[-\frac{z}{z-1}+\frac{z}{z-e^T}\right] = \\
&= -1+\frac{z-1}{z-e^T} = \frac{e^T-1}{z-e^T} \triangleq \frac{K_P}{z-u}.
\end{aligned}$$

We notice that $|u| > 1$, so it is an unstable pole.

1. To have the tracking of the step, the open loop transfer function $C\tilde{P}$ must include a pole in $z = 1$; therefore, $p = 1$ and the closed loop transfer function is

$$W(z) = \frac{K \cdot K_P}{(z-u)(z-1)+K \cdot K_P},$$



whose stability can be studied using the root locus of $D_W(z) = (z-u)(z-1) + K \cdot K_P$.

We see there is always at least one root outside the unit circle. Therefore, the system is always unstable, whatever $K \in \mathbb{R}$.

2. To obtain deadbeat tracking, the error $E(z)$ must be a polynomial in z^{-1}, i. e. of the type

$$E(z) = \frac{Q(z)}{z^q}, \qquad \text{where } q = \deg Q.$$

From the feedback interconnection, we know that $E(z)$ can be written as

$$E = R - Y = R - EC\tilde{P} \implies E = \frac{R}{1 + C\tilde{P}} = R \cdot (1 - W),$$

therefore we can write W in terms of Q, given the signal to track $R(z) = \frac{z}{z-1}$:

$$\frac{Q(z)}{z^q} = \frac{z}{z-1}(1 - W(z)) \implies W(z) = 1 - \frac{z-1}{z}\frac{Q(z)}{z^q} = \frac{z^{q+1} - Q(z)(z-1)}{z^{q+1}}.$$

Moreover, we have to check the relative degree of $W(z)$ to obtain $\operatorname{rdeg} C = 0$:

$$C(z) = \frac{W}{1-W}\frac{1}{\tilde{P}} \implies \operatorname{rdeg} C = \operatorname{rdeg} W - \cancel{\operatorname{rdeg}(1-W)} - \operatorname{rdeg} \tilde{P},$$

where $\operatorname{rdeg} \tilde{P} = 1$, while $\operatorname{rdeg}(1-W) = 0$ is required to have $W(\infty) \neq 1$, i. e. to obtain a causal controller C. Since we want $\operatorname{rdeg} C = 0$, then $\operatorname{rdeg} W = 1$.

Now it is matter to check what values of q allow to achieve stability, starting from the simplest one:

- case $q = 0$: Q is a zero degree polynomial $Q(z) = \alpha$, so

$$W(z) = \frac{z - \alpha(z-1)}{z} = \frac{z(1-\alpha) + \alpha}{z}.$$

  To satisfy the condition $\operatorname{rdeg} W = 1$, α must be 1 in order to cancel z from the numerator, yielding $W(z) = \frac{1}{z}$. However, this transfer function does not internally stabilize the system, because the unstable zero of $\tilde{P}$ is not cancelled.
- case $q = 1$: Q is a first degree polynomial $Q(z) = \alpha z + \beta$, so

$$W(z) = \frac{z^2 - (\alpha z + \beta)(z-1)}{z^2} = \frac{z^2(1-\alpha) + z(\alpha - \beta) + \beta}{z^2}.$$

  As before, to have $\operatorname{rdeg} W = 1$ we must choose $\alpha = 1$, cancelling z^2 from the numerator:

$$W(z) = \frac{z(1-\beta) + \beta}{z^2}$$

  To find β, we impose that the denominator $D_W - N_W$ of $\frac{W}{1-W}$ cancels the unstable pole $z = u$ of $\tilde{P}$:

$$(D_W - N_W)(u) = 0 \iff u^2 - u(1-\beta) - \beta = 0 \iff \beta = -\frac{u^2 - u}{u - 1} = -u.$$

Therefore, $W(z) = \frac{z(1+u)-u}{z^2}$ and the controller is

$$C(z) = \frac{W}{1-W}\frac{1}{\tilde{P}} = \frac{z(1+u) - u}{\underbrace{(z-1)\cancel{(z-u)}}_{z^2 - z(1+u) + u}} \cdot \frac{\cancel{z-u}}{K_P} = \frac{1}{K_P}\frac{z(1+u) - u}{z-1}.$$

3. To obtain the asymptotic tracking of the reference signal $r(t) = \cos t$, the poles of $R(z)$ must be included into $C\tilde{P}$. The poles of $R(z)$ are obtained by sampling and $\mathscr{Z}$-transformation:

$$\begin{aligned} R(z) &= \mathscr{Z}[\mathscr{S}_T[r(t)]] = \\ &= \mathscr{Z}[\cos(kT)] = \\ &= \mathscr{Z}\left[\frac{e^{kjT} + e^{-jkT}}{2}\right] = \\ &= \frac{1}{2}\frac{z}{z - e^{jkT}} + \frac{1}{2}\frac{z}{z - e^{-jkT}}, \end{aligned}$$

therefore the denominator of C must be of the type

$$\left(z - e^{jkT}\right)\left(z - e^{-jkT}\right) = z^2 - 2z\cos T + 1 = z^2 - z + 1.$$

In order to obtain the required relative degree $\operatorname{rdeg} C = 0$, the simplest controller has the form

$$C(z) = \frac{\alpha z^2 + \beta z + \gamma}{z^2 - z + 1},$$

which corresponds to a closed-loop transfer function

$$W(z) = \frac{C\tilde{P}}{1 + C\tilde{P}} = \frac{K_P(z-a)(z-b)}{(z^2 - z + 1)(z - u) + K_P(\alpha z^2 + \beta z + \gamma)}.$$

To have internal stability, the denominator D_W of $W(z)$ must be stable and the numerator of $C(z)$ cannot simplify the unstable pole of $\tilde{P}(z)$.

We have three degrees of freedom (α, β, γ) to place three stable roots of D_W wherever we like; for example, we can place them all in $z = 0$:

$$D_W(z) = z^3 - (u + 1 - K_P\alpha)z^2 + (u + 1 + K_P\beta)z - u + K_P\gamma = z^3 \implies$$

$$\implies \begin{cases} u + 1 - K_P\alpha = 0 \\ u + 1 + K_P\beta = 0 \\ -u + K_P\gamma = 0 \end{cases} \implies \begin{cases} K_P\alpha = u + 1 \\ K_P\beta = -(u+1) \\ K_P\gamma = u, \end{cases}$$

therefore the chosen controller is

$$C(z) = \frac{1}{K_P}\frac{(u+1)z^2 - (u+1)z + u}{z^2 - z + 1},$$

whose numerator does not simplify the unstable pole of $\tilde{P}(z)$, making the system also internally stable, because

$$(u+1)u^2 - (u+1)u + u = u^3 \neq 0.$$

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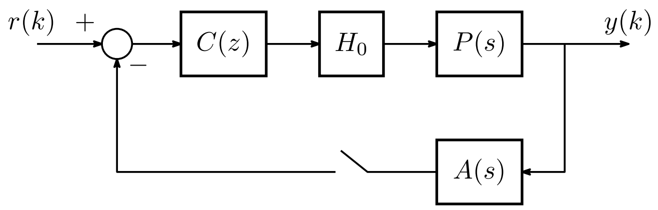
Exercise 2. Consider the interconnection shown in figure, with $T = 0.02\,\mathrm{s}$ and with

$$P(s) = \frac{5}{s+10}.$$

Assume that we need to insert in the scheme an antialiasing filter so that the sinusoidal signal of frequency $\omega \geq \Omega = \frac{2\pi}{T}$ are attenuated of a factor $a = 100$. Assume that $A(s)$ is a second order antialiasing filter. Design a PID continuous time controller using the techniques based on phase margin, in such a way that the closed loop system satisfies:

- tracking of the ramp reference signal with $e_r = 0.1$ asymptotic error;
- rise time $t_r = 0.2\,\mathrm{s}$ and overshoot $m_p = 0.05$.

Determine the corresponding discrete time transfer function $\tilde{C}(z)$ using the Euler backward method.



Solution. The controller must be such that the resulting system is type 1, and we are given three constraints (e_r, t_r, m_p). The type of the closed-loop system is related to the multiplicity of poles in the origin of $C(s)P(s)$, therefore a PID controller has to be chosen, because of its pole in the origin.

The three parameters K_I, T_I and T_D account for the other three constraints:

$$C(s) = \frac{K_I}{s}(1 + T_I s + T_I T_D s^2)$$

and to achieve properness we will need to add also a high frequency pole later on.

From the constraint on the tracking error e_r, we have

$$\mathbb{K}_\mathrm{B}[C(s)P(s)] = \frac{1}{e_r} \iff K_I \cdot \frac{5}{10} = \frac{1}{0.1} \iff K_I = 20.$$

To simplify the controller synthesis, we can embed the $\frac{K_I}{s}$ term directly into the process

$$P'(s) \triangleq \frac{K_I}{s} P(s) = \frac{100}{s(s+10)}$$

and design a controller $C'(s) = 1 + T_I s + T_I T_D s^2$ for this new process. This way, the controller required by the problem will be

$$C(s) = \frac{K_I}{s} C'(s).$$

From the time domain specifications, we obtain the desired crossover frequency and phase margin:

$$\omega_c^* \approx \frac{2}{t_r} = 10\,\mathrm{rad/s} \qquad m_\phi^* \approx 1.04 - 0.8 m_p = 1\,\mathrm{rad}.$$

At ω_c^*, the process $P'(s)$ has magnitude and phase

$$|P'(j\omega_c^*)| = \frac{100}{|j\omega_c^*| \cdot |10 + j\omega_c^*|} = \frac{100}{10 \cdot \sqrt{10^2 + 10^2}} = \frac{1}{\sqrt{2}} \simeq 0.707$$

$$\angle P'(j\omega_c^*) = -\frac{\pi}{2} - \arctan\frac{\omega_c^*}{10} = -\frac{3}{4}\pi \simeq -2.36\,\text{rad},$$

from which the phase margin is $m_\phi = \pi + \angle P'(j\omega_c^*) = \frac{\pi}{4} \simeq 0.785\,\text{rad}$.

To design the digital controller, we should take into account two phase contributions that slightly lower the available phase margin:

1. zero-order holder delay $\left(\sim e^{-s\frac{T}{2}}\right)$:

$$\phi_{\text{ZOH}} = \arctan\frac{\omega_c^* T}{2} \simeq 0.1\,\text{rad}$$

2. antialiasing filter phase contribution:

$$\phi_A \geq \frac{T}{2\pi} 4\xi\omega_c^*\sqrt{a} \simeq 0.90\,\text{rad}$$

   where $\xi = \frac{1}{\sqrt{2}}$ to have a monotonic frequency response.

The controller $C'(s)$ has to be such that $C'(j\omega_c^*) = Me^{j\Phi}$, where

$$M = \frac{1}{|P'(j\omega_c^*)|} = \sqrt{2} \simeq 1.41$$

$$\Phi = m_\phi^* - m_\phi + \phi_{\text{ZOH}} + \phi_A \simeq 1.215\,\text{rad}.$$

Now, the parameters of $C'(s)$ can be calculated solving the following equation:

$$C'(j\omega_c^*) = 1 + jT_I\omega_c^* - T_I T_D(\omega_c^*)^2 = Me^{j\Phi} \implies$$

$$\implies \begin{cases} \Re\{C(j\omega_c^*)\} = M\cos\Phi = 1 - T_I T_D(\omega_c^*)^2 \\ \Im\{C(j\omega_c^*)\} = M\sin\Phi = T_I\omega_c^* \end{cases} \implies \begin{cases} T_D = \dfrac{1 - M\cos\Phi}{\omega_c^* M\sin\Phi} \simeq 0.038\,\text{s} \\ \\ T_I = \dfrac{M\sin\Phi}{\omega_c^*} \simeq 0.133\,\text{s}. \end{cases}$$

The continuous-time PID controller is then

$$C(s) = \frac{K_I}{s}C'(s) = 20\,\frac{1 + 0.133s + 0.05s^2}{s(1 + \underbrace{0.001}_{T_H} s)}$$

where $1/T_H \gg \omega_c^*$ is a high-frequency pole to obtain properness of the controller, that was chosen to be $1/T_H = 100\omega_c^* = 1000\,\text{rad/s}$.

By applying the Euler backward transform

$$s \longleftrightarrow \frac{1 - z^{-1}}{T}$$

we obtain a discretized version of the controller:

$$\tilde{C}(z) = 50.54\,\frac{z^2 - 1.935z + 0.9423}{(z-1)(z-0.0476)}. \qquad \blacksquare$$

Exercise 3. Consider the difference equation

$$y(k) - ay(k-1) = u(k) - 2u(k-1)$$

where a is a real parameter. Determine the free response, the transfer function and the impulse response. Determine the asymptotic stability and the BIBO stability for every value of a.

Solution. From the $\mathscr{Z}$-transform of the difference equation, we get

$$\begin{aligned} y(k) - ay(k-1) &= u(k) - 2u(k-1) \\ &\Big\downarrow \mathscr{Z} \\ Y(z) - az^{-1}Y(z) - ay(-1) &= U(z) - 2z^{-1}U(z) - \cancel{2u(-1)} \\ (1 - az^{-1})Y(z) &= (1 - 2z^{-1})U(z) + ay(-1) \end{aligned}$$

where $u(-1) = 0$ because the input is always causal. The output signal $\mathscr{Z}$-transform can then be broken into a forced response $Y_f(z)$ and a free response $Y_\ell(z)$:

$$Y(z) = \underbrace{\overbrace{\frac{1 - 2z^{-1}}{1 - az^{-1}}}^{W(z)} \cdot U(z)}_{Y_f(z)} + \underbrace{\frac{ay(-1)}{1 - az^{-1}}}_{Y_\ell(z)},$$

therefore the free response is the antitransform of $Y_\ell(z)$:

$$Y_\ell(z) \xrightarrow{\mathscr{Z}^{-1}} y_\ell(k) = a^k \cdot \underbrace{ay(-1)}_{y(0)} = a^k y(0).$$

From the transfer function $W(z)$ we can derive the impulse response by antitransformation:

$$W(z) = \frac{z-2}{z-a} \implies \frac{W(z)}{z} = \frac{z-2}{z(z-a)} = \frac{A}{z} + \frac{B}{z-a} = \frac{(A+B)z - aA}{z(z-a)} \implies$$

$$\implies \begin{cases} A + B = 1 \\ -aA = -2 \end{cases} \implies \begin{cases} A = \frac{2}{a} \\ B = \frac{a-2}{a} \end{cases} \implies$$

$$\implies W(z) = \frac{2}{a} + \frac{a-2}{a} \cdot \frac{z}{z-a} \xrightarrow{\mathscr{Z}^{-1}} w(k) = \frac{2}{a}\delta(k) + \frac{a-2}{a} \cdot a^k.$$

Regarding stability, we have

- *asymptotic stability* iff

$$y_\ell(k) \xrightarrow{k \to +\infty} 0 \iff |a| < 1,$$

  i. e. when $W(z)$ is asymptotically stable;

- *BIBO stability* iff

$$y_f(k) \xrightarrow{k \to +\infty} 0 \iff |a| < 1 \land a = 2,$$

  i. e. when $W(z)$ is BIBO stable. ■

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