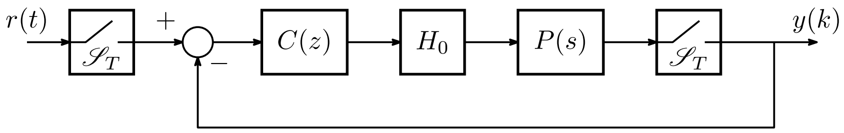
Exercise 1. Consider the scheme represented in the figure, where

$$P(s) = \frac{s-1}{s+1},$$

$\mathscr{S}_T$ is the sampler with sampling period T and H_0 is the zero-order holder.



1. Find, if possible, a digital controller $C(z) = \frac{K}{z-p}$, where $K, p \in \mathbb{R}$, that internally stabilizes the system and such that the output is able to track *asymptotically* the step reference signal.
2. Find, if possible, a digital controller with relative degree zero that internally stabilizes the system and such that the output is able to track *asymptotically* the step reference signal.
3. Find, if possible, a digital controller with relative degree one that internally stabilizes the system and such that the output is able to *deadbeat* track the step reference signal.

Solution. First of all, we calculate the discretized $\tilde{P}(z)$ by sample and hold:

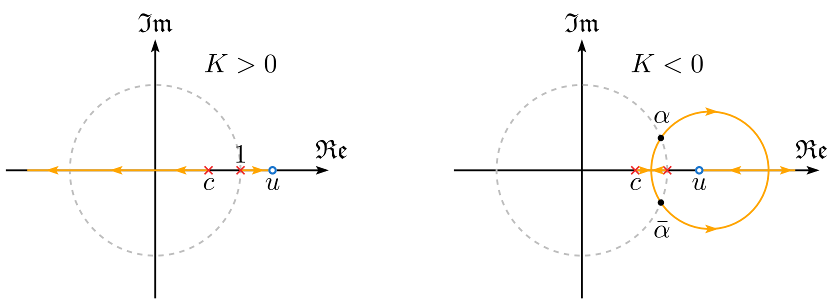
$$\begin{aligned}
\tilde{P}(z) &= \frac{z-1}{z}\mathscr{Z}\left[\mathscr{S}_T\left[\mathscr{L}^{-1}\left[\frac{P(s)}{s}\right]\right]\right] = \\
&= \frac{z-1}{z}\mathscr{Z}\left[\mathscr{S}_T\left[\mathscr{L}^{-1}\left[-\frac{1}{s} + \frac{2}{s+1}\right]\right]\right] = \\
&= \frac{z-1}{z}\mathscr{Z}\left[\mathscr{S}_T\left[-1 + 2e^{-t}\right]\right] = \\
&= \frac{z-1}{z}\mathscr{Z}\left[-1 + 2(e^{-T})^k\right] = \\
&= \frac{z-1}{z}\left[-\frac{z}{z-1} + 2\frac{z}{z-e^{-T}}\right] = \\
&= -1 + 2\frac{z-1}{z-e^{-T}} = \frac{z-(2-e^{-T})}{z-e^{-T}} \triangleq \frac{z-u}{z-c}.
\end{aligned}$$

We notice that $|u| > 1$ and so it is unstable, while $|c| < 1$ is stable.

1. To have the tracking of the step, the open loop transfer function $C\tilde{P}$ must include a pole in $z = 1$; therefore, $p = 1$ and the closed loop transfer function is

   $$W(z) = \frac{K(z-u)}{(z-c)(z-1) + K(z-u)},$$

   Stability can be studied using the root locus of $D_W(z) = (z-c)(z-1) + K(z-u)$:



Let $\alpha, \bar{\alpha}$ be the roots of $D_W(z) = z^2 - z(1 + c - K) + (c - Ku)$; then the zero-degree term $(c - Ku)$ is the product $\alpha\bar{\alpha} = |\alpha|^2$ and the value of K for which $|\alpha| = 1$ can be easily obtained:

$$c - Ku = 1 \iff K = \frac{c-1}{u} = -\frac{1 - e^{-T}}{2 - e^{-T}}.$$

Therefore, the system is stable for $-\frac{1-e^{-T}}{2-e^{-T}} < K < 0$. The system is also internally stable, as no unstable zero-pole cancellations occur between $\tilde{P}$ and C. The controller is

$$C(z) = \frac{K}{z-1}.$$

2. We can resort to the direct synthesis method. First, we check what relative degree must have $W(z)$:

$$C(z) = \frac{W}{1-W}\frac{1}{\tilde{P}} \implies \operatorname{rdeg} C = \operatorname{rdeg} W - \cancel{\operatorname{rdeg}(1-W)} - \cancel{\operatorname{rdeg}\tilde{P}},$$

where $\operatorname{rdeg} \tilde{P} = 0$ is given, while $\operatorname{rdeg}(1 - W) = 0$ is required to have $W(\infty) \neq 1$, i. e. to obtain a causal controller C. Since we require $\operatorname{rdeg} C = 0$, also $\operatorname{rdeg} W = 0$.

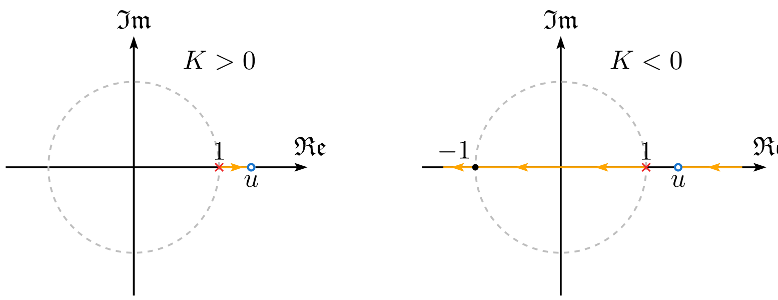
W should allow to stabilize the system, by cancelling the unstable zero of $\tilde{P}$, so it needs a numerator $N_W = X(z)(z-u)$. Moreover, to allow tracking the step signal, the denominator of $C\tilde{P}$ must be $D_W - N_W = Y(z)(z-1)$. In conclusion, W is of the form

$$W = \frac{X(z)(z-u)}{Y(z)(z-1) + X(z)(z-u)}$$

where, to obtain the required $\operatorname{rdeg} W = 0$, it need be $\deg X = \deg Y$. The simplest polynomials we can pick have degree zero:

$$X(z) = K, \quad Y(z) = 1 \implies W = \frac{K(z-u)}{z-1+K(z-u)}$$

and then we have to find the values of K that stabilize the system.



From the root locus of $z - 1 + K(z - u)$, it is clear that $K < 0$, and the value of K for which the branch crosses the unit circle is

$$-1 - 1 + K(-1-u) \implies K = -\frac{2}{u+1} = -\frac{2}{3 - e^{-T}},$$

therefore the system is stable for $-\frac{2}{3-e^{-T}} < K < 0$.

In conclusion, the controller is the following:

$$C(z) = \frac{K\cancel{(z-u)}}{z-1}\frac{z-c}{\cancel{z-u}} = K\frac{z-c}{z-1}.$$

3. To obtain deadbeat tracking, the error $E(z)$ must be a polynomial in z^{-1}, i.e. of the type

$$E(z) = \frac{Q(z)}{z^q}, \qquad \text{where } q = \deg Q.$$

From the feedback interconnection, we know that $E(z)$ can be written as

$$E = R - Y = R - EC\tilde{P} \implies E = \frac{R}{1 + C\tilde{P}} = R \cdot (1 - W),$$

therefore we can write W in terms of Q, given the signal to track $R(z) = \frac{z}{z-1}$:

$$\frac{Q(z)}{z^q} = \frac{z}{z-1}(1 - W(z)) \implies W(z) = 1 - \frac{z-1}{z}\frac{Q(z)}{z^q} = \frac{z^{q+1} - Q(z)(z-1)}{z^{q+1}}$$

Now it is matter to check what values of q allow to achieve stability, starting from the simplest one:

- case $q = 0$: Q is a zero degree polynomial $Q(z) = \alpha$, so

$$W(z) = \frac{z - \alpha(z-1)}{z} = \frac{z(1-\alpha) + \alpha}{z}.$$

  To satisfy the condition $\operatorname{rdeg} W = 1$, α must be 1 in order to cancel z from the numerator, yielding $W(z) = \frac{1}{z}$. However, this transfer function does not internally stabilize the system, because the unstable zero of $\tilde{P}$ is not cancelled.

- case $q = 1$: Q is a first degree polynomial $Q(z) = \alpha z + \beta$, so

$$W(z) = \frac{z^2 - (\alpha z + \beta)(z-1)}{z^2} = \frac{z^2(1-\alpha) + z(\alpha - \beta) + \beta}{z^2}.$$

  As before, to have $\operatorname{rdeg} W = 1$ we must choose $\alpha = 1$, cancelling z^2 from the numerator:

$$W(z) = \frac{z(1-\beta) + \beta}{z^2}$$

  To find β, we impose that the numerator of W cancels the unstable zero $z = u$ of $\tilde{P}$:

$$W(u) = 0 \iff u(1-\beta) + \beta = 0 \iff \beta = \frac{u}{u-1} = \frac{2 - e^{-T}}{1 - e^{-T}}.$$

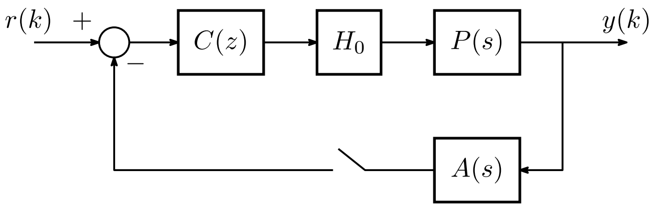
Therefore, $W(z) = \frac{z-u}{z^2(1-u)}$ and the controller is

$$C(z) = \frac{W}{1-W}\frac{1}{\tilde{P}} = \frac{\cancel{z-u}}{z^2(1-u) - z + u}\frac{z-c}{\cancel{z-u}} = \frac{z-c}{z^2(1-u) - z + u}. \qquad \blacksquare$$

Exercise 2. Consider the interconnection shown in figure, with $T = 0.01\,\mathrm{s}$ and with

$$P(s) = \frac{2}{s+1}.$$

Assume that we need to insert in the scheme an antialiasing filter so that the sinusoidal signal of frequency $\omega \geq \Omega = \frac{2\pi}{T}$ are attenuated of a factor $a = 10$. Assume that $A(s)$ is a second order antialiasing filter. Design a PD or a PI continuous time controller using the techniques based on phase margin, in such a way that the closed loop system satisfies the specifications on rise time $t_r = 0.2\,\mathrm{s}$, and on the overshoot $m_p = 0.05$. Determine the corresponding discrete time transfer function $\tilde{C}(z)$ using the MPZ method.



Solution. From the time domain specifications, we obtain the desired crossover frequency and phase margin:

$$\omega_c^* \approx \frac{2}{t_r} = 10\,\mathrm{rad/s} \qquad m_\phi^* \approx 1.04 - 0.8 m_p = 1\,\mathrm{rad}.$$

At ω_c^*, the process $P(s)$ has magnitude and phase

$$|P(j\omega_c^*)| = \frac{2}{|1 + j\omega_c^*|} = \frac{2}{\sqrt{1+10^2}} \simeq 0.199$$

$$\angle P(j\omega_c^*) = -\arctan \omega_c^* \simeq -1.47\,\mathrm{rad},$$

from which the phase margin is $m_\phi = \pi + \angle P(j\omega_c^*) \simeq 1.67\,\mathrm{rad}$.

To design the digital controller, we should take into account two phase contributions that slightly lower the available phase margin:

1. zero-order holder delay $\left(\sim e^{-s\frac{T}{2}}\right)$:

$$\phi_{\text{ZOH}} = \arctan \frac{\omega_c^* T}{2} \simeq 0.10\,\mathrm{rad}$$

2. antialiasing filter phase contribution:

$$\phi_A \geq \frac{T}{2\pi} 4\xi\omega_c^* \sqrt{a} \simeq 0.14\,\mathrm{rad}$$

   where $\xi = \frac{1}{\sqrt{2}}$ to have a monotonic frequency response.

The controller $C(s)$ has to be such that $C(j\omega_c^*) = M e^{j\Phi}$, where

$$M = \frac{1}{|P(j\omega_c^*)|} \simeq 5.02$$

$$\Phi = m_\phi^* - m_\phi + \phi_{\text{ZOH}} + \phi_A \simeq -0.48\,\mathrm{rad}$$

To decrease the phase, a PI controller is required:

$$C(s) = K_P \left(1 + \frac{1}{sT_I}\right),$$

where the parameters can be obtained from the previous specifications at ω_c^*:

$$C(j\omega_c^*) = K_P \left(1 - j\frac{1}{\omega_c^* T_I}\right) = Me^{j\Phi} \implies$$

$$\implies \begin{cases} \Re\{C(j\omega_c^*)\} = M\cos\Phi = K_P \\ \Im\{C(j\omega_c^*)\} = M\sin\Phi = -\dfrac{K_P}{\omega_c^* T_I} \end{cases} \implies \begin{cases} K_P = M\cos\Phi \simeq 4.45 \\ T_I = -\dfrac{1}{\omega_c^* \tan\Phi} \simeq 0.19\,\text{s}. \end{cases}$$

The continuous-time PI controller is then

$$C(s) = K_P \frac{s + \frac{1}{T_I}}{s} = 4.45\,\frac{s+5.2}{s} \triangleq \frac{C_0(s)}{s}.$$

Applying the MPZ method, we have to find the gain K_d of the equivalent discrete-time controller

$$\tilde{C}(z) = K_d \frac{z - e^{-5.2T}}{z-1} \simeq K_d \frac{z-0.95}{z-1} \triangleq \frac{\tilde{C}_0(z)}{z-1}$$

such that its Bode gain $\mathbb{K}_\mathrm{B}[\tilde{C}(z)]$ is equal to the one of $C(s)$ multiplied by T^ν, where $\nu = 1$ is the multiplicity of poles in the origin:

$$\begin{aligned} &\mathbb{K}_\mathrm{B}[\tilde{C}(z)] = T^\nu \cdot \mathbb{K}_\mathrm{B}[C(s)] \\ \iff &\tilde{C}_0(1) = T \cdot C_0(0) \\ \iff &K_d(1-0.95) = 0.01 \cdot 4.45 \cdot 5.2 \\ \iff &K_d \simeq 4.57. \end{aligned}$$

Therefore, the digital controller is

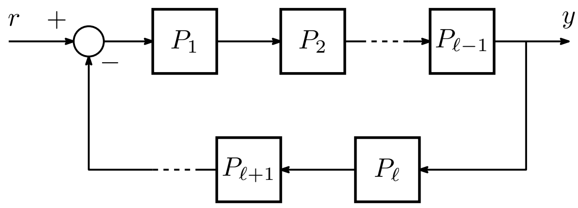
$$\tilde{C}(z) \simeq 4.57\,\frac{z-0.95}{z-1}. \qquad \blacksquare$$

Exercise 3. Recall the definition of internal stability of an interconnection and give the conditions ensuring that a feedback interconnection is internally stable.

Definition. An interconnection is internally stable if every transfer function

$$W_{i,j}(z) \triangleq \frac{Y_i(z)}{N_j(z)}$$

of the system is BIBO-stable $\forall i, j$.



Theorem. An interconnected feedback system is internally stable iff

$$\begin{cases} N_1 N_2 \cdots N_p + D_1 D_2 \cdots D_p & \text{is Schur stable} \\ \deg(N_1 N_2 \cdots N_p + D_1 D_2 \cdots D_p) = \deg(D_1 D_2 \cdots D_p), \end{cases}$$

where $P_\ell(z) \triangleq \frac{N_\ell(z)}{D_\ell(z)}$ is a coprime representation of $P_\ell(z)$ $\forall \ell$. ■

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